From 3 nucleons to 3 quarks

Yu.A.Simonov

Abstract

Some short history of few-body methods originated from the famous Skornyakov-Ter-Martirosyan equation is given, including latest development of Faddeev formalism and Efimov states. The 3q system is shown to require an alternative, which is provided by the hyperspherical method (K-harmonics) which is highly successful for baryons.

1 Introduction

The Skornyakov and Ter-Martirosyan paper [1] which appeared in 1956 marked the beginning of a new era in few-body physics, when a somewhat neglected part of nuclear physics was promoted to the successful domain of theoretical physics. As a result the few-body science has become a field accumulating fast developing methods: L.D.Faddeev generalized the Skornyakov-Ter-Martirosyan Equation (STME) [2] and has given a rigorous mathematical foundation for the theory of 3 particles [3], many numerical methods have been introduced, for a review and references see [4]. As an immediate consequence of STME, a new effect was found in 1971, called the Efimov effect [5], which is studied till now with respect to possible experimental consequences [6].

The STME and Faddeev technic is most useful when particles are nearly on-shell, so that e.g. 3-body results do not depend much on the potential shape, but rather described by the on-shell two-body t-matrix as it is for the quartet $n-d$ scattering. The bound states of tritium and ^3He provide another example, where the interaction at small distances (far off- shell) is important. To treat such systems an alternative method - the Hyperspherical Formalism (HF) (or K-harmonics method) was developed and the system of the Schroedinger-like equations was written [7]. Its development was marked with many successful applications both in nuclear and atomic physical see e.g. [8, 9]. Recently it was understood that HF is probably the best suitable for systems with confinement such as 3 quarks, where the interaction is a three-body one, and confining so that the t-matrix formalism cannot be applied. Accuracy of HF as applied to the 3q system was found to be remarkably good [10, 11] allowing for the 1% bias in the baryon mass [12].

This talk is intended to demonstrate the physics of the 3-body system, and a qualitative analysis of two alternative approaches discussed above.

2 The STME and Faddeev approach

In the system of 3 equal-mass particles with arbitrary numeration one can introduce the total kinetic energy E and the momentum $\mathbf{k}$ in the pair (2,3) and the relative momentum $\mathbf{p}$ of particle 1, namely $\mathbf{p} = \frac{\mathbf{k}_2+\mathbf{k}_3}{3} - \frac{2}{3}\mathbf{k}_1$, $\mathbf{k} = \frac{\mathbf{k}_2-\mathbf{k}_3}{2}$.

The symmetric function of the ground state Ψ_{symm} is expressed through partial w.v.

$$\Psi_{symm} = \psi(\mathbf{k}_{23}, \mathbf{p}_1) + \psi(\mathbf{k}_{31}, \mathbf{p}_2) + \psi(\mathbf{k}_{12}, \mathbf{p}_3) \tag{1}$$

with the normalization condition

$$\int |\Psi_{symm}|^2 d^3\mathbf{k} d^3\mathbf{p} = 1. \tag{2}$$

It is convenient to extract the free 3-body Green's function, introducing

$$\psi(\mathbf{k}, \mathbf{p}) = \frac{\chi(\mathbf{k}, \mathbf{p})}{\mathbf{k}^2 + \frac{3}{4}\mathbf{p}^2 - mE} \tag{3}$$

and the 3-body rescattering equation, equivalent to the summing the "bridge" Feynman diagrams (nonrelativistic) is [2]

$$\chi(\mathbf{k}, \mathbf{p}) = \chi_0(\mathbf{k}, \mathbf{p}) - 2\int m \frac{t(k, |\frac{\mathbf{p}}{2} + \mathbf{p}'|, E - \frac{3}{4}\frac{p^2}{m})\chi(|\mathbf{p} + \frac{\mathbf{p}'}{2}|, p')d\mathbf{p}'}{\mathbf{p}'^2 + \mathbf{p}\mathbf{p}' + \mathbf{p}^2 - mE}. \tag{4}$$

Here $t(k, k', \varepsilon)$ is the 2-body t-matrix, representing the "knot" in a bridge diagram, and

$$\chi_0(\mathbf{k}, \mathbf{p}) = -2m \frac{t(k, \frac{\mathbf{p}}{2} + \mathbf{p}_0, E - \frac{3}{4}\frac{p^2}{m})\varphi_\alpha(\mathbf{p} + \frac{1}{2}\mathbf{p}_0)}{\mathbf{p}^2 + \mathbf{p}_0^2 + \mathbf{p}\mathbf{p}_0 - mE}, \tag{5}$$

where φ_α is the 2-body bound state, while $\mathbf{p}_0$ is the momentum of incident particle.

Near the bound-state pole t-matrix can be written as

$$t(k,k',\varepsilon) = \frac{g(k\varepsilon)g(k',\varepsilon)}{(2\pi)^2 m(\alpha + i\sqrt{2m\varepsilon})} + O(r_0) \tag{6}$$

where $\alpha = 1/a, a$ is the scattering length and $g(k,\varepsilon)$ formfactor, $g(0,0) = 1$ and $g(k,\varepsilon)$ fast decreases when $k \sim 1/r_0$ and $\varepsilon \sim \frac{1}{mr_0^2}$.

Let us assume now that the range of integration in (4) is small $p, p' \ll 1/r_0$. Then one can insert (6) in (4) with $g \cong 1$, and one gets- for the 3-body bound-state w.f.

$$(\alpha - \sqrt{\frac{3}{4}\mathbf{p}^2 - E})\chi(\mathbf{k},\mathbf{p}) + 8\pi \int \frac{d\mathbf{p}'}{(2\pi)^3} \frac{\chi(|\mathbf{p} + \frac{\mathbf{p}'}{2}|, p')}{\mathbf{p}'^2 + \mathbf{p}\mathbf{p}' + \mathbf{p}^2 - mE} = 0. \tag{7}$$

This is the STME for a 3-body bound state. The off-shell generalization of STME is the Faddeev equation(4). As it was correctly stated in [1], the bound-state equation (7) cannot be used for tritium and ^3He, since it has no lower bound for energy due to the Thomas theorem [13]. This can be easily understood rewriting (7) in the form $\chi = \int K\chi d\mathbf{p}'$, and calculating the norm of $K, \parallel K \parallel = \int d\mathbf{p}d\mathbf{p}(K(\mathbf{p},\mathbf{p}'))^2$, which diverges logarithmically at large momenta, implying that there are formally infinitely many bound states. The physical situation corresponds to the cut-off form-factors $g(k,\varepsilon)$ present in K, which leads to the finite result for the norm $\parallel K \parallel$.

A specific situation occurs when the 2-body scattering length a is large, $a \gg r_0$. Then the number of bound states lying between $\left(-\frac{1}{ma^2}\right)$ and $\left(-\frac{1}{mr_0^2}\right)$ is approximately equal to

$$N \sim \frac{1}{\pi} \ln \frac{|a|}{r_0} \tag{8}$$

and when $|a|$ is increasing, $|a| \to \infty$, there appears an accumulation point of bound states (the Efimov effect[5]). For 3 nucleons however $N < 1$ and the effect is absent, but for three ^4He atoms $a = 104\mathring{A}, r_0 \cong 7\mathring{A}$ and the effect is theoretically possible [6].

Since the Efimov states are almost on-shell, it is convenient to calculate them using the 3-body unitarity and the N/D method [14]. Numerical results obtained (see Fig. 7 of [14]) support the estimate (8) and yield the explicit position of levels near the energy threshold.

To conclude with the bound state equation (7) it is interesting to study the properties of the bound wave function, e.g. the size of the bound system. Here one encounters an important difference between 2- and 3-body systems [15]. Namely the 2-body loosely bound system with a small binding energy ε, $m\varepsilon r_0^2 \ll 1$ has a radius of the order of $r_2 = \frac{1}{\sqrt{m\varepsilon}}, r_2 \gg r_0$. For the 3-body system the situation may be twofold. In case when a bound 2-body system exists as a subsystem and the 3-body bound state is close to the 2+1 threshold, one has a quasi-two-body situation, whereas when 2-body bound subsystems are absent the size of the 3-body bound state is always r_0 however small binding energy is [16]. Modern calculations of 3-body bound states in the framework of STME and its development – Faddeev equations are done for ^3H and ^3He using systems of around 30 equations and exploiting realistic potentials describing NN scattering and bound states in the large energy interval (0-350 MeV). see e.g. the review in [17]. Unfortunately results of calculations yield significant underbinding of around 10-15%, may be due to three-body forces, which are not exactly known, hence final results are model dependent, see [18] for an example and references.

We go now to the dN scattering which was also the topic of the primary paper [1]. The corresponding equations look like

$$
\begin{aligned}
\frac{(\sqrt{3k^2/4-ME}-\alpha_t)}{k^2-k_0^2} a_{3/2}(\mathbf{k},\mathbf{k}_0) &= \frac{-1}{k_0^2+k^2+\mathbf{k}\mathbf{k}_0-ME} - \\
&- \int \frac{4\pi a_{3/2}(\mathbf{k}',\mathbf{k}_0)}{(k^2+k^2+\mathbf{k}\mathbf{k}'-ME)(k^2-k_0^2)} \frac{d\mathbf{k}'}{(2\pi)^3}; \\
\frac{(\sqrt{3k^2/4-ME}-\alpha_t)}{k^2-k_0^2} a_{3/2}(\mathbf{k},\mathbf{k}_0) &= \frac{1/2}{k_0^2+k^2+\mathbf{k}\mathbf{k}_0-ME} + \\
&+ \int \frac{4\pi\{1/2a_{1/2}(\mathbf{k}',\mathbf{k}_0)+3/2b_{1/2}(\mathbf{k}',\mathbf{k}_0)\}}{(k^2+k^2+\mathbf{k}\mathbf{k}'-ME)(k^2-k_0^2)} \frac{d\mathbf{k}'}{(2\pi)^3}, \\
\frac{(\sqrt{3k^2/4-ME}-\alpha_t)}{k^2-k_0^2} b_{1/2}(\mathbf{k},\mathbf{k}_0) &= \frac{3/2}{k_0^2+k^2+\mathbf{k}\mathbf{k}_0-ME} + \\
&+ \int \frac{4\pi\{3/2a_{1/2}(\mathbf{k}',\mathbf{k}_0)+1/2b_{1/2}(\mathbf{k}',\mathbf{k}_0)\}}{(k^2+k^2+\mathbf{k}\mathbf{k}'-ME)(k^2-k_0-)} \frac{d\mathbf{k}'}{(2\pi)^3}.
\end{aligned}
\tag{9}
$$

Here $a_{3/2}$ is the Nd quartet ($S = 3/2$) scattering amplitude, while $b_{1/2}$ and $a_{1/2}$ are doublet ($S = \frac{1}{2}$) amplitudes corresponding to the singlet and triplet last NN interaction respectively. It is seen that the kernel for $S = 3/2$ is mostly negative and allows for a faster convergence, in contrast to the doublet ($S = 1/2$) case. Numerical result for quartet scattering length $a_{3/2} = 5.1$fm obtained in [1] is not far from experimental value [19], whereas doublet scattering requires full off-shell calculation [4].

3 Hyperspherical Method

Heretofore the basic dynamics was assumed to be quasi-two-body (however the Faddeev technic allows for the full off-shell description), in the sense that typical distance R between an interacting pair and a third spectator particle is large, $R \gg r_0$. However this situation is an exclusion, and not the rule, which can be understood from the representation of the w.f. through the 3 body Green's function ($\boldsymbol{\xi}, \boldsymbol{\eta}$ are Jacobi coordinates)

$$\psi(\boldsymbol{\xi}, \boldsymbol{\eta}) = \psi_0(\boldsymbol{\xi}, \boldsymbol{\eta}) + \int G(\boldsymbol{\xi} - \boldsymbol{\xi}', \boldsymbol{\eta} - \boldsymbol{\eta}') V_3(\boldsymbol{\xi}', \boldsymbol{\eta}) \psi(\boldsymbol{\xi}'., \boldsymbol{\eta}') d\boldsymbol{\xi}' d\boldsymbol{\eta}'. \tag{10}$$

Here $G(\xi, \boldsymbol{\eta}) = \frac{K_2(\kappa\rho)}{\rho^2}$, $\kappa = \sqrt{2m|E|}$, $\rho = \sqrt{\boldsymbol{\xi}^2 + \boldsymbol{\eta}^2}$, and V_3 includes all interaction terms. The asymptotics of ψ is given by G and is equal to

$$\psi(\boldsymbol{\xi}, \boldsymbol{\eta}) \sim \frac{1}{\rho^4}, \quad \rho \to \infty. \tag{11}$$

Hence the 3-body kynematics tends to concentrate all 3-body w.f. inside the interaction region of all 3 particles which generates small radius of w.f. even for barely bound 3-body states. (This is also true for N -body systems $N \geq 3$). In this situation any pair angular momentum l_{ij} contributes to the total energy of the system an amount $\Delta E \sim \frac{l_{ij}(l_{ij}+1)}{2mr_0^2}$ which for the 3 nucleon system with $r_0 \sim 1$ fm and for $l_{ij} = 1$ is of the order of $\Delta E \sim 50$ MeV, while for the $3q$ system with $m = m_q \sim 0.3$ GeV and $r_0 \sim 0.5$ fm, $\Delta E_q \sim 600$ MeV.

Therefore it is advantageous to have a wave function with the minimal number of pair internal angular momenta for the given total momentum L. This basis is provided by hyperspherical functions (K - harmonics) due to the following properties:

i) the solution of the condition $\hat{l}_{ij}\Psi = 0, i \neq j = 1, ...N$ is given by the representation $\Psi_{K=0} = \Psi_0(\rho)$,

$$\rho^2 = \frac{1}{N} \sum_{i<j=1}^{n} (\mathbf{r}_i - \mathbf{r}_j)^2 \tag{12}$$

where all particles are assumed to have the same mass.

ii) The function $\Psi_K(\mathbf{r}_i, ...\mathbf{r}_N) = u_K(\Omega)\chi_K(\rho)$, where $u_K(\Omega) = \frac{\mathcal{P}_K(\mathbf{r}_i, ...\mathbf{r}_N)}{\rho^K}$ and $\mathcal{P}_K$ - harmonic polynomial contains excited angular momenta $l_1, ...l_{N-1}$ the arithmetic sum of which is equal to K.

Therefore the basis Ψ_K corresponds to the minimal excitation of angular momenta and is advantageous for compact N-body systems. Since as was explained the majority of such systems are compact, the Hyperspherical Expansion Approach (HEA) [7] formulated as a system of coupled integral or differential equations has proved to be very successful both for few-nucleon systems [7],[8], where short-range correlations can be taken into account in the hyperspherical correlated basis (last ref. in [18]), and atomic physics [9]. It was understood afterwards [10],[11] that the HEA works even better for $3q$ systems, since interaction there contains no repulsive core and confinement excludes two-body channels.

Therefore already the lowest approximation with $K = 0$ yields the 1% accuracy for the baryon energy [10, 11, 12].

In this case the baryon state is characterized by the grand angular momentum K and radial quantum number $n = 0, 1, 2$, which counts number of zeros of the w.f. in the ρ-space. A typical calculation was done in [12], and the result depends on only two input parameters: string tension $\sigma = 0.15$ GeV^2 and $\alpha_s = 0.4$, while current masses of light quarks have been put to zero. The spin-averaged masses $\frac{1}{2}(M_n + M_\Delta)$ have been computed to eliminate effect of hyperfine splitting.

To illustrate the simplicity of the method, let us quote the the equation for the for the dominant hyperspherical harmonics $\psi_K(\rho) = \frac{\chi_K(\rho)}{\sqrt{\rho}}$,

$$-\frac{1}{2\mu}\frac{d^2\psi_K}{d\rho^2} + W_{KK}(\rho)\psi_K(\rho) = E_K\psi_K(\rho) \tag{13}$$

where $W_{KK}(\rho)$ is the sum of kinetic (angular) and potential energies, and μ is a constituent quark mass to be found below dynamically. The nonrelativistic appearance of this equation contains nevertheless the full relativistic dynamics, since μ is the einbein field needed to get rid of square roots in the relativistic quark action.

The explicit expression for W_{KK} is

$$W_{KK}(\rho) = \frac{d}{2\mu\rho^2} + V_{KK}(\rho), \quad d = (K + \frac{3}{2})(K + \frac{5}{2}), \tag{14}$$

while $V_{KK}(\rho) = (u_K^+(\Omega)\hat{V}u_K(\Omega))$, is the total potential $\hat{V}$, including 2-body and 3-body parts, averaged over hyperspherical harmonics, which is done analytically. E.g. for the Y-type $3q$ confining potential one has $V_{KK}(\rho) =$

$1.58\sigma\rho$. It is remarkable that to find the eigenvalues E_K with the 1% accuracy one does not need to solve equation (13), but instead is approximating $W_{KK}(\rho)$ near the minimum point ρ_0 by the oscillator well:

$$W_{KK}(\rho) = W_{KK}(\rho_0) + \frac{1}{2}(\rho - \rho_0)^2 W''_{KK}(\rho_0), \quad \frac{dW_{KK}}{d\rho}|_{\rho=\rho_0} = 0. \tag{15}$$

The resulting eigenvalues are found immediately:

$$E_{Kn} \cong W_{KK}(\rho_0) + \omega(n + \frac{1}{2}), \quad \omega^2 = W''_{KK}/\mu. \tag{16}$$

The total baryon mass is calculated as $M_{Kn}(\mu) = \frac{3}{2}\mu + E_{Kn}(\mu)$, and finally $\mu = \mu_0$ is to be found from the stationary point condition $\frac{\partial M_{Kn}(\mu)}{\partial \mu}|_{\mu=\mu_0} = 0$. This gives the constituent quark mass $\mu_0 = 0.957\sqrt{\sigma} = 0.37$ GeV and finally the baryon mass is $M_{Kn}(\mu_0)$. The masses $\frac{1}{2}(M_n + M_\Delta)$ computed in this way are shown below in Table 1.

Table 1

Baryon masses (in GeV) averaged over hyperfine spin splitting for $\sigma = 0.15$ GeV^2, $\alpha_s = 0.4$, $m_i = 0$.

State	$M_{Kn} + \langle \Delta H \rangle_{self}$	$\langle \Delta H \rangle_{coul}$	M^{tot}_{Kn}	M^{tot}(exp)
$K = 0, n = 0$	1.36	-0.274	1.08	1.08
$K = 0, n = 1$	2.19	-0.274	1.91	?
$K = 0, n = 2$	2.9	-0.274	2.62	?
$K = L = 1, n = 0$	1.85	-0.217	1.63	1.6
$K = 2, n = 0$	2.23	-0.186	2.04	?

As it seen from the Table the calculated spin-averaged mass $\frac{1}{2}(M_N + M_\Delta)$ agrees well with the experimental average, the same is also true for lowest negative parity states with $K = L = 1$, which should be compared with $\frac{1^-}{2}, \frac{3^-}{2}$ states of N and Δ respectively.

We also notice that breathing modes ($n > 0$) have excitation energy around 0.8 GeV while orbital excitations $K = L = 1$ have energy interval around 0.5 GeV.

One of important advantages of HEA is that in the lowest approximation there is no need for numerical computations – as demonstrated above the result for the mass can be obtained analytically with 1% accuracy as can be checked by comparison with exact calculations, see [10]-[12].

To conclude, the on-shell approach of STME (and its Faddeev generalization) and the HEA are two alternatives which describe opposite physical situations. Their coexistence has played a very important stimulating role for the development of the few-body physics in the last four decades.

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